1

10 ème Cours
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<u>Recall</u>: $C|\mathbb{Q}_p$, $F = C^\flat$, $\infty \in |X|$

$H/\mathcal{O}_C$ p-divisible group

$\mathbb{H}/\overline{\mathbb{F}}_p$ $\quad \mathbb{H} \underset{\overline{\mathbb{F}}_p}{\otimes} \mathcal{O}_C/p\mathcal{O}_C \longrightarrow H \underset{\mathcal{O}_C}{\otimes} \mathcal{O}_C/p\mathcal{O}_C$

quasi-isogeny

$(D, \varphi) = \mathbb{D}(\mathbb{H})$ Covariant isocrystal

$\mathrm{Fil}\, D_C$ = Hodge filtration given by H

$D_C / \mathrm{Fil}\, D_C = \mathrm{Lie}\, H[\frac{1}{p}]$

<u>Then</u>:

$$0 \to V_p(H) \otimes \mathcal{O}_X \to \mathcal{E}(D, p^{-1}\varphi) \longrightarrow i_{\infty *} \mathrm{Lie}\, H[\tfrac{1}{p}] \to 0$$

$\Rightarrow$ if $u : \mathcal{E}(D, p^{-1}\varphi) \twoheadrightarrow i_{\infty *} D_C$

$u^{-1}\left(i_{\infty *} \mathrm{Fil}\, D_C\right)$ is a trivial vector bundle

Lubin-Tate space

$\mathbb{H} / \overline{\mathbb{F}}_p$ 1-dimensional formal p-divisible group of height n.

$\mathcal{X} = \mathrm{Def}(\mathbb{H}) \simeq \mathrm{Spf}\left(W(\overline{\mathbb{F}}_p)[[x_1, \dots, x_{n-1}]]\right)$

$\mathcal{X}_\eta \simeq \mathring{\mathbb{B}}^{n-1}_{\breve{\mathbb{Q}}_p}$ $\quad\quad \breve{\mathbb{Q}}_p = W(\overline{\mathbb{F}}_p)[\frac{1}{p}]$

$\downarrow \pi_{dR}$ Gross-Hopkins period morphism

$\mathbb{P}^{n-1}_{\breve{\mathbb{Q}}_p}$ p-adic analog of Griffiths period morphism

$(D, \varphi) = \mathbb{D}(\mathbb{H})$ then $x \in \mathcal{X}(\mathcal{O}_C) = \mathcal{X}_\eta(C)$

$\pi_{dR}(x) = \mathrm{Fil}\, D_C \subset D_C$ Codimension 1 Hodge filtration

Th (Lafaille / Gross-Hopkins):

π_{dR} is surjective étale cover.

↑ Grothendieck-Messing def. theory (p-adic analog of Kodaira-Spencer)

i.e. $\forall C$ $\forall \operatorname{Fil} D_C$ codimension 1 subspace of D_C, $\operatorname{Fil} D_C$ is the Hodge filtration of a lift of $\mathbb{H}$ to $\mathcal{O}_C$.

→ explicit computation using quasi-logarithms

Coro. $\forall$ any deg. 1 modification of $\mathcal{O}_X(\frac{1}{n})$, $\mathcal{E}$,

$$0 \to \mathcal{E} \to \mathcal{O}_X\left(\frac{1}{n}\right) \to \mathcal{F} \to 0$$

degree 1 torsion coherent sheaf

$\mathcal{E}$ is trivial, $\mathcal{E} \simeq \mathcal{O}_X^n$.

Dual statement:

Prop: $0 \to \mathcal{O}_X^n \to \mathcal{E} \to \overset{\text{deg. 1 torsion coherent}}{\mathcal{F}} \to 0$ a deg. 1 modification of $\mathcal{O}_X^n$. Then $\mathcal{E} \simeq \mathcal{O}_X\left(\frac{1}{d}\right) \oplus \mathcal{O}_X^{n-d}$ for some $1 \leq d \leq n$.

→ the modification is given by a surjection

$$C(-1)^n = \left(t^{-1} B^+_{dR} / B^+_{dR}\right)^n \twoheadrightarrow \underset{\text{1-dim } C\text{-v.s.}}{L}$$

Up to replace $\mathcal{O}_X^n$ by $\mathcal{O}_X^{n-i}$ and $\mathcal{E}$ by $\mathcal{E}'$ with $\mathcal{E} = \mathcal{E}' \oplus \mathcal{O}_X^i$ one can suppose $\mathbb{Q}_p(-1)^n \hookrightarrow L$ i.e. is given by an element of $\Omega(C) \subset \mathbb{P}^{n-1}(C)$.

$D = D_{\frac{1}{n}}$ division algebra invariant $\frac{1}{n}$.

$D = \mathrm{End}\left(\mathcal{O}_X\left(\frac{1}{n}\right)\right)$ induces $\left(D^{opp}\right)^\times_X \xrightarrow{\sim} \underline{\mathrm{Aut}}\left(\mathcal{O}_X\left(\frac{1}{n}\right)\right)$

as group schemes $/X$

$\Rightarrow (D^{op})^\times$-torsors on $X \simeq GL_n$-torsors on X

$\rotatebox{90}{$\simeq$}$ v.b. of rk. n/X

$\big\lfloor (D^{op})^\times_X$ = pure inner form of GL_n

$$0 \to \mathcal{O}_X^n \to \mathcal{E} \to \mathcal{F} \to 0$$

$\Updownarrow$ via this equivalence

$$0 \to \mathcal{O}_X(-\tfrac{1}{n}) \to \mathcal{E}' \to \mathcal{F}' \to 0 \quad \text{sequence of v.b. } \circlearrowleft D^{op}$$

$\Updownarrow$ dual modification

$$0 \to \mathcal{E}'' \to \mathcal{O}_X(\tfrac{1}{n})^n \to \mathcal{F}'' \to 0 \quad \text{sequence of v.b. } \circlearrowleft D$$

$\rotatebox{90}{$\simeq$}$ $\mathcal{E}'^{\vee}$

Drinfeld: Any element of $\Omega(C)$ is the Hodge filtration of a special formal $\mathcal{O}_D$-module

$\Rightarrow \mathcal{E}''$ trivial $\simeq D \otimes_{\mathbb{Q}} \mathcal{O}_X$ = Tate module

$\Rightarrow \mathcal{E} \simeq \mathcal{O}_X(\tfrac{1}{n})$.

$\big\lfloor$ reverse everything

Proof of the classification for rk. 2 vector bundles

Prop: $\mathcal{F}$ degree 1 torsion coherent / X

(1) $0 \to \mathcal{E} \to \mathcal{O}(d_1) \oplus \mathcal{O}(d_2) \to \mathcal{F} \to 0$ $\qquad d_1, d_2 \in \mathbb{Z}$

$$d_1 \neq d_2 \Rightarrow \mathcal{E} \simeq \begin{cases} \mathcal{O}(d_1 - 1) \oplus \mathcal{O}(d_2) \\ \mathcal{O}(d_1) \oplus \mathcal{O}(d_2 - 1) \end{cases}$$

(2) $0 \to \mathcal{E} \to \mathcal{O}(d)^2 \to \mathcal{F} \to 0$ $\qquad d \in \mathbb{Z}$

$$\mathcal{E} \simeq \mathcal{O}(d - \tfrac{1}{2}) \quad \text{or} \quad \mathcal{E} \simeq \mathcal{O}(d-1) \oplus \mathcal{O}(d)$$

(3) $0 \to \mathcal{E} \to \mathcal{O}(d + \tfrac{1}{2}) \to \mathcal{F} \to 0$ $\qquad d \in \mathbb{Z}$

$$\mathcal{E} \simeq \mathcal{O}(d)^2$$

$\to$ (1) by explicit computation

(2) Consequence of Lubin-Tate case

(3) " $\qquad$ " Drinfeld case

$\mathcal{E}$ any vector bundle $/X$, $\infty \in |X|$

Then for $d >> 0$, $\exists$ modification

$$0 \to \mathcal{E} \to \mathcal{O}_X(d)^n \to \underbrace{\mathcal{T}_\infty}_{\text{torsion support at } \infty} \to 0$$

$\to$ induction on $n = \mathrm{rk}(\mathcal{E})$.

$$\begin{tikzcd}
0 \arrow[r] & \mathcal{E}' \arrow[r] \arrow[d, "\text{modif. at } \infty"] & \mathcal{E} \arrow[r] \arrow[d] & \overbrace{\mathcal{L}}^{\text{line bundle}} = \mathcal{O}_X(d) \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & \mathcal{O}_X(d')^{n-1} \arrow[r] & \mathcal{E}'' \arrow[r] & \mathcal{O}_X(d) \arrow[r] & 0
\end{tikzcd}$$

induction hypothesis

One can choose $d' \geq d \Rightarrow \mathrm{Ext}^1(\mathcal{O}(d), \mathcal{O}(d')) = 0$

$\hookrightarrow \mathcal{O}_X(d') \hookrightarrow \mathcal{O}_X(d'+1)$

$$\Rightarrow \mathcal{E}'' \simeq \mathcal{O}_X(d) \oplus \mathcal{O}_X(d')^{n-1} \xhookrightarrow[\text{modification at } \infty]{} \mathcal{O}_X(d')^n \quad \square$$

One concludes in the following way:

$\mathcal{E}$ rk. 2 v.b. $/X$

$$0 \to \mathcal{E} \to \mathcal{O}_X(d)^2 \xrightarrow{u} \mathcal{F} \to 0$$

$\mathrm{Fil}^\cdot \mathcal{F} =$ filtration of $\mathcal{F}$

s.t. $\mathrm{gr}^i \mathcal{F} = \begin{cases} 0 \\ \text{degree 1 torsion at } \infty \end{cases}$

$$\mathrm{Fil}^\cdot \mathcal{O}_X(d)^2 \; \cancel{\mathrm{Fil}^\cdot} = u^{-1}\left(\mathrm{Fil}^\cdot \mathcal{F}\right)$$

$$\mathrm{Fil}^n \mathcal{O}_X(d)^2 = \begin{cases} \mathcal{O}_X(d)^2 & \text{for } n \ll 0 \\ \mathcal{E} & \text{for } n \gg 0 \end{cases}$$

$$\mathrm{Fil}^{n+1} \mathcal{O}_X(d)^2 = \begin{cases} \mathrm{Fil}^n \mathcal{O}_X(d)^2 \\ \text{or a degree 1 modification of } \mathrm{Fil}^n \mathcal{O}_X(d)^2 \end{cases}$$

Conclusion by induction $\square$

Weakly admissible implies admissible

$E = \mathbb{Q}_p$.

$K | \mathbb{Q}_p$ discrete valuation perfect residue field

$K_0 = W(k_K)_{\mathbb{Q}} \circlearrowleft \sigma$ $\qquad C = \hat{\overline{K}}$, $G_K = \mathrm{Gal}(\overline{K}|K)$

$$\varphi\text{-}\mathrm{ModFil}_{K|K_0} = \{ (\underbrace{D, \varphi}_{K_0\text{-isocrystal}}, \underbrace{\mathrm{Fil}^\bullet D_K}_{\text{"Hodge" filtration of } D \otimes_{K_0} K}) \}$$

$t_N =$ "$v_p(\det \varphi)$"

$t_H = \sum_{i \in \mathbb{Z}} i . \dim gr^i D_K$

$$V_{cris}(D, \varphi, \mathrm{Fil}^\bullet D_K) = \mathrm{Fil}^0 (D \otimes_{K_0} B_{cris})^{\varphi = Id}$$

$$= \mathrm{Fil}^0 (D \otimes_{K_0} B[\tfrac{1}{t}])^{\varphi = Id} \circlearrowleft G_K$$

Def: $(D, \varphi, \mathrm{Fil}^\bullet D_K)$ is admissible if

$$\dim_{\mathbb{Q}_p} V_{cris}(D, \varphi, \mathrm{Fil}^\bullet D_K) = \dim_{K_0} D$$

$V_{cris} : \varphi\text{-ModFil}^{ad}_{K/K_0} \xrightarrow{\sim} \mathrm{Rep}^{cris}_{\mathbb{Q}_p} G_K$

with inverse given by D_{cris}.

Def: $(D, \varphi, \mathrm{Fil}^{\cdot} D_K)$ is weakly admissible if

$$t_H(D, \varphi, \mathrm{Fil}^{\cdot} D_K) = t_N(D, \varphi, \mathrm{Fil}^{\cdot} D_K)$$

and $\forall D' \subset D$ sub-isocrystal

$$t_H(D', \varphi_{|D'}, \mathrm{Fil}^{\cdot} D_K \cap D'_K) \leq t_N(D', \varphi_{|D'}, \mathrm{Fil}^{\cdot} D_K \cap D'_K)$$

Th (Colmez-Fontaine): Weakly admissible $\Leftrightarrow$ Admissible.

$\underset{\text{easy}}{\Leftarrow}$

Reinterpretation in terms of semi-stability.

$\deg = t_H - t_N$

$\mathrm{rk} = \dim_{K_0} D$. $\underset{\text{easy}}{\rightsquigarrow}$ $\boxed{\varphi\text{-ModFil}^{wa}_{K/K_0} = \varphi\text{-ModFil}^{ss,0}_{K/K_0}}$

$\mu = \frac{\deg}{\mathrm{rk}}$

semi-stable slope 0.

* $X \ni \infty$ G_K stabilizes ∞.

$$\begin{array}{ccc}
\varphi\text{-ModFil}_{K/K_0} & \longrightarrow & \left\{ (\mathcal{E}, \Lambda) \,\middle|\, \begin{array}{l} \mathcal{E}\ G_K\text{-equivariant v.b.}/X \\ \Lambda \subset \widehat{\mathcal{E}}_\infty[\tfrac{1}{t}]\ G_K\text{-invariant lattice} \end{array} \right\} \\
(D, \varphi, \mathrm{Fil}^\bullet D_K) & \longmapsto & (\mathcal{E}(D,\varphi), \mathrm{Fil}^0(D \otimes B_{dR})) \\
 & & \downarrow \\
 & & \{ G_K\text{-equivariant v.b.}/X \}
\end{array}$$

T

$\downarrow$

$\mathcal{E}(D, \varphi, \mathrm{Fil}^\bullet D_K)$

$$\deg\left(\mathcal{E}(D,\varphi,\mathrm{Fil}^\bullet D_K)\right)$$

$$= \underbrace{\deg\left(\mathcal{E}(D,\varphi)\right)}_{-t_N(D,\varphi)} + \underbrace{\left[\mathrm{Fil}^0 D\otimes B_{dR} : D\otimes B_{dR}^+\right]}_{t_H(D,\varphi,\mathrm{Fil}^\bullet D_K)}$$

$$= \deg\left(D,\varphi,\mathrm{Fil}^\bullet D_K\right).$$

$$H^0\left(X,\mathcal{E}(D,\varphi,\mathrm{Fil}^\bullet D_K)\right) = V_{\mathrm{cris}}\left(D,\varphi,\mathrm{Fil}^\bullet D_K\right)$$

Classification of v.b. theorem $\Rightarrow$ [$\mathcal{E}$ v.b./X is semi-stable slope 0 $\Leftrightarrow$ $\dim_{\mathbb{Q}_p} H^0(X,\mathcal{E}) = \mathrm{rk}\,\mathcal{E}$]

7

Thus for $A = (D, \varphi, \mathrm{Fil}^\bullet D_K)$

A admissible $\Leftrightarrow$ $\mathcal{E}(A)$ semi-stable slope 0

A weakly admissible $\Leftrightarrow$ $\mu(\mathcal{E}(A)) = 0$ and $\forall B \subset A$ strict subobject $\mu(\mathcal{E}(B)) \leq 0$

~~Test~~ semi-stability on sub-isocrystals.

$\rightarrow$ we already see from this that admissible $\Rightarrow$ weakly admissible.

* $\mathcal{E}(A) = G_K$- equivariant vector bundle

Prop: Sub-objects of A (strict) $\simeq$ G_K-invariant sub-objects of $\mathcal{E}(A)$.

$\rightarrow$

In fact, if $B_e = B[\frac{1}{t}]^{\varphi = Id}$ P.I.D.

G

G_K

[$\varphi\text{-Mod}_{K_0} \hookrightarrow \mathrm{Rep}_{B_e}(G_K)$ = Continuous representations of G_K with values in a B_e-module free of finite rank.

$(D, \varphi) \longmapsto \left(D \underset{K_0}{\otimes} B[\frac{1}{t}]\right)^{\varphi = Id}$

= abelian category

Since ∞ = only point of X with finite G_K-orbit.

= G_K-equivariant coherent sheaves on $X \setminus \{\infty\}$

$\mathcal{E}(D,\varphi)|_{X \setminus \{\infty\}}$

Easy: Sub-Isocrystals of ~~$(D \otimes B[\frac{1}{t}])^{\varphi = Id}$~~ (D, φ)

$\Updownarrow$

Sub-representations of $\left(D \otimes B[\frac{1}{t}]\right)^{\varphi = Id}$

Inverse sends $M \longmapsto \left(\left(M \underset{B_e}{\otimes} B[\frac{1}{t}]\right)^{G_K}, Id \otimes \varphi\right)$

Proof of w.a. $\Rightarrow$ a: A w.a. Then

the Harder-Narasimhan filtration of $\mathcal{E}(A)$

is G_K-invariant since unique.

$\Rightarrow$ Comes from a filtration of A

This is trivial since A is semi-stable.

Thus $\mathcal{E}(A)$ is semi-stable. $\square$